ON THE SIMILARITY OF HYPERSONIC VISCOUS FLOWS AROUND SLENDER BODIES

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The equations of the laminar boundary layer on sharp flat and axisymmetric bodies are considered for high flight Mach numbers M. For the flat plate, similar equations have been studied by Shen [1] and Lees [2]. Criteria for the extent of the influence of the boundary layer on the outer inviscid flow are established for $M \gg 1$. For thin bodies, conditions for the similarity of the flows are determined.

1. Let (x, y, ϕ) be the curvilinear coordinates, where y represents the distance normal to the body, x the distance from the nose along the body surface, and ϕ the meridional angle. Let u and v designate the x and y components of the velocity, respectively, r the radial distance from the axis of symmetry, and R the radius of curvature of the body profile $r = r_w(x)$. The components of the velocity-deformation and stress tensors then have the form (3), (4):

$$e_{xx} = \frac{R}{R+y}\left(\frac{\partial u}{\partial x} + \frac{v}{R}\right), \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{\varphi\varphi} = \frac{u \sin\theta + v\cos\theta}{r}$$

$$e_{xy} = \frac{1}{2}\frac{R}{R+y}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{R+y}{R}\frac{\partial}{\partial y}\left(\frac{Ru}{R+y}\right) \qquad (1.1)$$

$$e_{x\varphi} = e_{y\varphi} = 0, \qquad \sin\theta = \frac{\partial r}{\partial x}, \qquad \cos\theta = \frac{\partial r}{\partial y}$$

$$p_{xx} = -p + \lambda \operatorname{div} \mathbf{V} + 2\mu e_{xx}, \qquad p_{yy} = -p + \lambda \operatorname{div} \mathbf{V} + 2\mu e_{yy}$$

$$p_{\varphi\varphi} = -p + \lambda \operatorname{div} \mathbf{V} + 2\mu e_{\varphi\varphi}, \qquad p_{xy} = \mu e_{xy}, \qquad p_{x\varphi} = p_{y\varphi} = 0$$

Here $\mathbf{V}$ is the velocity vector, and μ and λ are the coefficients of viscosity. Denoting the unit vectors in the directions of x, y, and ϕ by $\mathbf{k}_x$, $\mathbf{k}_y$, and $\mathbf{k}_\phi$, we obtain:

$$\mathbf{p}_x = \mathbf{k}_x p_{xx} + \mathbf{k}_y p_{xy}, \quad \mathbf{p}_y = \mathbf{k}_x p_{xy} + \mathbf{k}_y p_{yy}, \quad \mathbf{p}_\varphi = \mathbf{k}_\varphi p_{\varphi\varphi}, \quad \mathbf{V} = \mathbf{k}_x u + \mathbf{k}_y v$$

$$\frac{\partial \mathbf{k}_x}{\partial x} = -\frac{\mathbf{k}_y}{R}, \quad \frac{\partial \mathbf{k}_x}{\partial \varphi} = \mathbf{k}_\varphi \sin\theta, \quad \frac{\partial \mathbf{k}_x}{\partial y} = \frac{\partial \mathbf{k}_y}{\partial y} = \frac{\partial \mathbf{k}_\varphi}{\partial x} = \frac{\partial \mathbf{k}_\varphi}{\partial y} = 0 \tag{1.2}$$

$$\frac{\partial \mathbf{k}_y}{\partial x} = \frac{\mathbf{k}_x}{R}, \quad \frac{\partial \mathbf{k}_y}{\partial \varphi} = \mathbf{k}_\varphi \cos\theta, \quad \frac{\partial \mathbf{k}_\varphi}{\partial \varphi} = -\mathbf{k}_x \sin\theta - \mathbf{k}_y \cos\theta$$

The laws of conservation of momentum, energy and mass, applied to a differential fluid element, yield the equations:

$$\rho r\left(1+\frac{y}{R}\right)\frac{d\mathbf{V}}{dt} = \frac{\partial}{\partial x}(\mathbf{p}_x r) + \frac{\partial}{\partial y}\left[r\left(1+\frac{y}{R}\right)\mathbf{p}_y\right] + \frac{\partial}{\partial \varphi}\left[\left(1+\frac{y}{R}\right)\mathbf{p}_\varphi\right] \tag{1.3}$$

$$\rho r\left(1+\frac{y}{R}\right)\frac{d}{dt}\left(\varepsilon + \frac{\mathbf{V}\cdot\mathbf{V}}{2}\right) = r\left(1+\frac{y}{R}\right)\operatorname{div}\left(\frac{\mu}{\sigma}\operatorname{grad} i\right) + \frac{\partial}{\partial x}(r\,\mathbf{p}_x\cdot\mathbf{V}) +$$
$$+ \frac{\partial}{\partial y}\left[r\left(1+\frac{y}{R}\right)\mathbf{p}_y\cdot\mathbf{V}\right] + \frac{\partial}{\partial \varphi}\left[\left(1+\frac{y}{R}\right)\mathbf{p}_\varphi\cdot\mathbf{V}\right] \tag{1.4}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\mathbf{V} = \frac{\partial \rho}{\partial t} + \frac{R}{r(R+y)}\left\{\frac{\partial}{\partial x}(r\rho u) + \frac{\partial}{\partial y}\left[r\left(1+\frac{y}{R}\right)\rho v\right]\right\} = 0 \tag{1.5}$$

In these equations ρ, ϵ, i, and σ represent density, internal energy, enthalpy, and Prandtl number respectively.

With the aid of (1.3), (1.5) and the obvious equality

$$\frac{d\varepsilon}{dt} = \frac{di}{dt} - \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho}\operatorname{div}\mathbf{V}$$

the equation (1.4) can be transformed into

$$\rho r\left(1+\frac{y}{R}\right)\frac{di}{dt} = \rho r\left(\frac{R+y}{R}\frac{\partial i}{\partial t} + u\frac{\partial i}{\partial x} + v\frac{R+y}{R}\frac{\partial i}{\partial y}\right) = r\left(1+\frac{y}{R}\right)\frac{dp}{dt} +$$
$$+ pr\left(1+\frac{y}{R}\right)\operatorname{div}\mathbf{V} + r\left(1+\frac{y}{R}\right)\operatorname{div}\left(\frac{\mu}{\sigma}\operatorname{grad} i\right) + r\,\mathbf{p}_x\cdot\frac{\partial \mathbf{V}}{\partial x} +$$
$$+ r\left(1+\frac{y}{R}\right)\mathbf{p}_y\cdot\frac{\partial \mathbf{V}}{\partial y} + \left(1+\frac{y}{R}\right)\mathbf{p}_\varphi\cdot\frac{\partial \mathbf{V}}{\partial \varphi} \tag{1.6}$$

Substitution of (1.1) and (1.2) into (1.3) and (1.5) leads to the equations of motion of a viscous, heat-conducting gas in curvilinear coordinates in an expanded scalar form. In the limit $r \to \infty$, these equations transform into the equations of two-dimensional flow. Let the thickness ratio and length of the body be of the order of β and l, respectively, and let the region in which viscous effects are significant have the dimension δ. Then,

$$x \sim l, \qquad y \sim \delta, \qquad u = V_\infty, \qquad v \sim V_\infty \delta / l, \qquad r \sim \beta l + \delta$$

Subscripts ∞ and w refer to the values in the undisturbed stream and at the body surface respectively.

We will assume that $\delta/l \ll 1$, $l/R \ll 1$. Then, using customary estimates of relative magnitudes and neglecting terms of the order $(\delta/l + l/R)(\delta/l)$, the equations (1.3)-(1.6) of motion are transformed into:

$$\rho r^\nu\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-r^\nu\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(r^\nu\mu\frac{\partial u}{\partial y}\right) \tag{1.7}$$

$$\rho r^\nu\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{u^2}{R}\right)=-r^\nu\frac{\partial p}{\partial y}+0\left(r^\nu\mu\frac{U_\infty}{\delta l}\right) \tag{1.8}$$

$$\rho r^\nu\left(u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}\right)=r^\nu u\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(r^\nu\frac{\mu}{\sigma}\frac{\partial i}{\partial y}\right)+r^\nu\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{1.9}$$

$$\frac{\partial(r^\nu\rho u)}{\partial x}+\frac{\partial(r^\nu\rho v)}{\partial y}=0 \tag{1.10}$$

Here $\nu = 0$ corresponds to the two-dimensional case and $\nu = 1$ to the axisymmetric case. We will postulate

$$\frac{\mu}{\mu_\infty}=C\left(\frac{i}{i_\infty}\right)^n,\qquad \frac{\rho}{\rho_\infty}=\frac{p}{p_\infty}\frac{i_\infty}{i},\qquad \sigma=\text{const} \tag{1.11}$$

We will assume that cooling, if any, will not change the order of magnitude of the temperature in the boundary layer.* Then, designating the order of magnitude of the shock angle at the nose by α, in this layer we will have

$$i\sim i_\infty\frac{\varkappa-1}{2}M_\infty^2,\quad \mu\sim\mu_\infty C\left(\frac{\varkappa-1}{2}M_\infty^2\right)^n,\quad p\sim p_\infty M_\infty^2\alpha^2,\quad \rho\sim\frac{2}{\varkappa-1}\rho_\infty\alpha^2 \tag{1.12}$$

Let ψ_0 represent the flux of mass crossing the plane $x = l$, which is intercepted by the bow shock wave, and ψ_w the influx of mass into the boundary layer. Then

$$\frac{\psi_0}{\rho_\infty V_\infty}\sim(\alpha l)^{\nu+1},\quad \frac{\psi_w}{\rho V_\infty}\sim[(\delta+l\beta)^{\nu+1}-(l\beta)^{\nu+1}],\quad \frac{\psi_w}{\psi_0}\sim\frac{2}{\varkappa-1}\alpha^{1-\nu}\frac{\delta}{l}\left(\frac{\delta}{l}+\beta\right)^\nu$$

Clearly, whenever $\delta/1 \ll 1$, the ratio ψ_w/ψ_0 is small and there must exist a region of flow where the influence of viscosity becomes insignificant.** It follows therefore that

$$\alpha\sim 1/M_\infty+\delta/l+\beta$$

* When $\sigma = 1$, $M_\infty \gg 1$ and $i_w = 0$ in the two-dimensional case

$$\frac{i}{i_\infty}\sim\frac{\varkappa-1}{2}M_\infty^2\frac{u}{V_\infty}\left(1-\frac{u}{V_\infty}\right)\quad\text{or}\quad\frac{i}{i_\infty}\sim\frac{\varkappa-1}{2}M_\infty^2$$

** In the two-dimensional case this fact has been established by Stewartson [5] on the basis of the solution of equations (1.7)-(1.9).

On the basis of (1.12), the inertial and viscous terms in (1.7) are of the order of

$$\frac{2}{\varkappa-1}\frac{\rho_\infty V_\infty^2}{l}\alpha^2(\delta+\beta l)^\nu \sim \frac{2p(\delta+l\beta)^\nu}{(\varkappa-1)C}, \qquad \left(\frac{\varkappa-1}{2}\right)^n \frac{V_\infty C\mu_\infty M_\infty^{2n}}{\delta^2}(\delta+\beta l)^\nu$$

respectively. In the viscous layer the ratio of these terms must be of the order of unity so that

$$\frac{\delta}{l}\sim\left(\frac{\varkappa-1}{2}\right)^{{}^1/{}_2(n+1)}\frac{M^n{}_\infty C^{1/2}}{N_{\mathrm{Re}}{}^{1/2}(1/M_\infty+\beta+\delta/l)}, \quad N_{\mathrm{Re}}=\frac{\rho_\infty V_\infty l}{\mu_\infty} \tag{1.13}$$

From (1.8) it then follows

$$\frac{1}{p}\frac{\partial p}{\partial y}\sim\frac{1}{(\varkappa-1)l}\left(\frac{\delta}{l}+\frac{l}{R}\right), \qquad \frac{\Delta p}{p}\sim\left(\frac{\delta}{l}+\frac{l}{R}\right)\frac{\delta}{(\varkappa-1)l}$$

where Δp is the pressure drop across the boundary layer.

In this manner, equation (1.8) can be altogether neglected in the viscous region and equations (1.7), (1.9), and (1.10) will represent Prandtl's equations for compressible gas generalized to the case of axisymmetric flow. When $\delta/r_w \ll 1$, we may set $r = r_w(x)$, and these equations take on the generally accepted form (6) with $\nu = 1$.

Let us introduce the parameter $K = \delta/\beta l$, which characterizes the relative influence of the boundary layer on the flow in the inviscid region. (When $K \ll 1$ the influence of the boundary layer is negligible in comparison with the influence of the body itself; when $K \gg 1$ the boundary layer plays the dominant role in shaping the outer flow.)

It follows from (1.13) that

$$K\sim\frac{\chi}{M_\infty\beta(1+M_\infty\beta+M_\infty\beta K)}, \qquad \chi=\left(\frac{\varkappa-1}{2}\right)^{{}^1/{}_2(n+1)}M_\infty^{2+n}\left(\frac{C}{N_{\mathrm{Re}}}\right)^{1/2} \tag{1.14}$$

In this manner, the influence of the boundary layer on the outer flow depends only on the relationship between the parameters* χ and $M_\infty\beta$. Through equation (1.14) the cases

$$\chi\ll M_\infty\beta(1+M_\infty\beta), \quad \chi\sim M_\infty\beta(1+M_\infty\beta), \quad \chi\gg M_\infty\beta(1+M_\infty\beta)$$

correspond uniquely to the cases

$$K\ll 1, \quad K\sim 1, \quad K\gg 1$$

* The present parameter χ differs from that in [1] and [2] only by a constant multiplier.

Simultaneously, the condition $\delta/l \ll 1$ will automatically be satisfied for arbitrary bodies in the first case, and for thin bodies for the other two cases. For the case $K \gg 1$, the parameter K together with $M_\infty \beta$ no longer represents the determining factor and is replaced by $K_1 = M_\infty \delta/l$ because it is the interaction between the boundary layer and the Mach cone which emerges as the dominant flow phenomenon.

From (1.14) it follows that $K_1 \quad \chi/(1 + K_1)$ and with $\delta/l \ll 1$ we must have $K_1 \ll M_\infty$. This last case includes the flow around the flat plate [2], [5] when $M_\infty \beta \ll 1$.

Analysis of all possible combinations leads to the following general formula, which contains as special cases those just discussed:

$$\frac{\delta}{l} \sim \beta_1 = \chi [M_\infty (1 + M_\infty \beta + \chi^{1/2})]^{-1}$$

In the inviscid region the flow is described by equation (1.8) without the viscous terms, by equation (1.10) and by the adiabatic energy equation in which [7] we set $u = V_\infty$ when the body is slender.

2. As in Section 1, let us assume that the flow between the body and the bow wave is divided into the viscous and the inviscid regions by the sharp boundary $r = r_\delta(x)$, which must be determined by the simultaneous solution of the appropriate equations in both regions. This latter assumption is well borne out when $M_\infty \gg 1$. For instance, for the flat plate (cone) with $\sigma = 1$ and $\kappa = 1.4$

$$y\left(\frac{p_\infty V_\infty}{M_\infty x}\right)^{1/2} = \text{const}\left[\xi + \ 0.48 M_\delta^2 - 0.34 M_\delta^2 \left(1 - \frac{5 i_w}{M_\delta^2}\right)\right]$$

Here ξ is the Blasius variable and the subscript refers to the quantities at the dividing surface. The edge of the boundary layer corresponds to $\xi = 4.3 \quad 5.3$, where $u/u_s = .97 \quad .995$. It is clear that changes of ξ within these bounds correspond to small changes in y with the order of magnitude given by $\Delta y/y \quad z/(\kappa - 1) M_\infty^2$. Analogous results were obtained by Stewartson [5] for the case of pressure variation $p \quad Q_0 x^{-1/2} + Q_1 + O(x^{1/2})$.

Let us introduce the dimensionless quantities

$$p_0 = \frac{p}{\rho_\infty V^2 (\beta_1 + \beta)^2} \qquad v_0 = \frac{v}{V_\infty (\beta_1 + \beta)}$$

$$y_0 = \frac{y}{l(\beta_1 + \beta)}, \quad r_0 = \frac{y + r_w}{l(\beta_1 + \beta)}, \quad x_0 = \frac{x}{l}$$

Also, let us set $\hat{\rho} = \rho/\rho_\infty$ and $T_1 = T/T_\infty$ in the inviscid region, and $u_2 = u/v_\infty$ and $i_2 = 2i/(\kappa - 1) i_\infty M_\infty^2$ in the viscous region. The equations then become

$$\rho_1 \frac{\partial v_0}{\partial x_0} + \rho_1 v_0 - \frac{\partial v_0}{\partial y_0} \frac{d^2 r_{w0}}{d x_0^2} = - \frac{\partial p_0}{\partial y_0}, \quad \frac{\partial}{\partial x_0} \frac{p_0}{\rho_1^{\varkappa}} + v_0 \frac{\partial}{\partial y_0} \frac{p_0}{\rho_1^{\varkappa}} = 0 \tag{2.1}$$

in the inviscid region, and

$$\begin{gathered}
\varkappa r_0^{\nu} \frac{p_0}{i_2}\left(u_2 \frac{\partial u_2}{\partial x_0} + v_0 \frac{\partial u_2}{\partial y_0}\right) = \frac{\varkappa - 1}{2} r_0^{\nu} \frac{\partial p_0}{\partial x_0} + A \frac{\partial}{\partial y_0}\left(r_0^{\nu} i_2 \frac{\partial u_2}{\partial y_0}\right) \\
\varkappa r_0^{\nu} \frac{p_0}{i_2}\left(u_2 \frac{\partial i_2}{\partial x_0} + v_0 \frac{\partial i_2}{\partial y_0}\right) = (\varkappa - 1) r_0^{\nu} u_2 \frac{\partial p_0}{\partial x_0} + \frac{1}{\sigma} A \frac{\partial}{\partial y_0}\left(r_0^{\nu} i_2 \frac{\partial i_2}{\partial y_0}\right) + 2 A i_2 \left(\frac{\partial u_2}{\partial y_0}\right)^2 \\
(A = \chi^2 / \theta^4, \quad \theta = M_\infty (\beta_1 + \beta))
\end{gathered} \tag{2.2}$$

in the viscous region.

The continuity equation remains unaltered.

The boundary conditions at the shock wave have the form

$$\begin{gathered}
p_0 = \frac{2}{\varkappa + 1} r_{*}'^2 - \frac{(\varkappa - 1)}{(\varkappa + 1)} \frac{1}{\varkappa \theta^2}, \qquad \rho_1 = \frac{(\varkappa + 1) \theta^2 r_{*0}'^2}{2 + (\varkappa - 1) \theta^2 r_{*0}'^2} \\
T_1 = \frac{\varkappa \theta^2 p_0}{\rho_1}, \quad v_0 = \frac{2}{\varkappa + 1} \frac{\theta^2 r_{*0}'^2 - 1}{\theta^2 r_{*0}'^2} - r_{w0}' \quad \text{at} \quad r_0 = r_{*0}(x_0)
\end{gathered} \tag{2.3}$$

where $r_*(x)$ represents the shape of the shock wave.

At the dividing surface between the two regions, $r_0 = r_{0\delta}(s_0)$, we set

$$u_2 \approx 1, \quad v_{01} = v_{02}, \quad i_2 = 2T_1 / (\varkappa - 1) M_\infty^2 \quad \text{at} \quad r_0 = r_{0\delta}(x_0) \tag{2.4}$$

Here v_{01} and v_{02} correspond to the velocity components in the inviscid and viscous regions respectively. At the body surface dimensionless boundary conditions read:

$$u_2 = v_0 = 0, \quad i_2 = i_w(x_0) \ \text{or} \ \frac{\partial i_2}{\partial y_0} = \lambda(x_0) \quad \text{at} \quad y = 0 \tag{2.5}$$

With the exception of the last of conditions (2.4), all the equations and all the boundary conditions contain only the two flow parameters χ and $M_\infty \beta$ (in addition to the dimensionless body profile and wall temperature variation) because θ itself depends on χ and $M_\infty \beta$. The last conditions (2.4) contains M_∞ alone. However, this condition states that

$$i_2 = \frac{2T_1}{(\varkappa - 1) M_\infty^2} \sim \frac{1}{(\varkappa - 1) M_\infty^2} + (\beta_1 + \beta)^2 \quad \text{at} \quad r_0 = r_{0\delta}(x_0) \tag{2.6}$$

Within the boundary layer $i_2 \approx 1$. Therefore, the decisive role in the determination of the temperature profile, and hence of the other viscid and inviscid quantities, is played by the processes of dissipation and heat conduction, and with respect to the overall formulation of the problem the last of conditions (2.4) is immaterial.

We can thus put forward the following similarity rule for the flow of a viscous heat-conducting perfect gas around slender bodies at $M_\infty \gg 1$.

For bodies with similar shape $r_0(x_0) = r_w(x)/\beta l$ and similar dimensionless wall temperature distributions $i_w(x_0)$ or $\lambda(x_0)$ in (2.5), the dimensionless quantities p_0, ρ_1, v_0, u_0 are functions only of the dimensionless coordinates x_0 and y_0, and of the parameters χ and $M_\infty \beta$. Furthermore, the dimensionless shock shape $r_{*0}(x_0)$ and edge of the boundary layer $r_{0\delta}(x_0)$ depend only on χ and $M_\infty \beta$.

When the flow field about an arbitrary body is known for a given set of conditions, it is possible to compute the flow field under different conditions around a body obtained by affine transformation from the original body subject to the conditions $M_\infty \beta$ = const, χ = const.

The known [8] hypersonic similarity rule for inviscid flows emerges as a special case of the proceding rule when $\chi \ll M_\infty \beta (1 + M_\infty \beta)$ i.e. when $\beta_1 \ll \beta$.

When $\beta_1 \gg \beta$, the parameter $M_\infty \beta$ becomes unimportant and the flow is determined by χ.

It should be noted that the similarity in the boundary layer is disturbed near its outer boundary where $[(\kappa - 1)/2] M_\infty^2 (1 - u^2/u_\delta^2)$ 1, and consequently the ratio T/T_δ is of the order of unity (even though it exceeds unity in magnitude). In this region we cannot neglect condition (2.6). Furthermore, the ratio $(\partial p/\partial y)/(\partial p/\partial y)_1$ (where the subscript 1 refers to the inviscid region) is on the order of T_δ/T 1. Strictly speaking, this means that the system of equations of the boundary layer should be supplemented by equation (1.8) without its viscous term. However, as discussed earlier, the relative size of this zone is small, and hence its influence on the overall flow is unimportant.

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